UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3801**

ASSESSMENT PATTERN : **MATH3801A**

MODULE NAME : **Logic**

DATE : **27-May-11**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH3801A-001-EXAM-62

TURN OVER

All questions may be attempted, but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the set of formulae, $\mathcal{L}$, in the first order predicate language.

   (b) Define the degree of a formula.

   (c) Define the weight of an element of $\mathcal{L}_{string}$.

   (d) Prove that, if α is a formula, then the weight of α is -1.

   (e) Consider the following elements of $\mathcal{L}_{string}$, where x, y are variables, P is a unary predicate, and Q is a binary predicate:

   (i) $\neg y \forall x P x$

   (ii) $\Rightarrow \neg Q x x \forall y P y$

   (iii) $\forall x \Rightarrow \Rightarrow P x Q x y$

   For each string above, determine whether or not it is a formula, justifying your answer.

2. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.

   (b) Define what it means to say that a proposition is a tautology.

   (c) Use the semantic tableaux method to determine whether or not each of the following propositions is a tautology (where α, β, γ are primitive propositions). If a proposition is not a tautology, describe a valuation for which it fails to be true.

   (i) $((\neg\alpha) \Rightarrow \beta) \Rightarrow ((\neg\beta) \Rightarrow \alpha)$

   (ii) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\alpha \Rightarrow (\gamma \Rightarrow \beta))$

   (iii) $(\alpha \Rightarrow (\beta \vee \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma))$

3. (a) Define what it means to have a proof of a proposition α from a set of propositions S, giving also the set of axioms and the rule of deduction.

   (b) State and prove the Deduction Theorem for propositional logic.
   (You may assume the validity of the following theorem: $\vdash (\alpha \Rightarrow \alpha)$)

   (c) Use the Deduction Theorem to show the following:
   $$\{\alpha \Rightarrow \beta\,,\, \beta \Rightarrow \gamma\,,\, \gamma \Rightarrow \delta\} \vdash (\alpha \Rightarrow \delta)$$

4. (a) Define an $\mathcal{L}(\Pi, \Omega)$-structure, where Π is a given set of predicate symbols and Ω is a given set of functional symbols, with assigned arities.

   (b) Describe a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure U is a (normal) model of the theory if and only if U is a group of order 5.

   (c) Suppose that S is a set of sentences in a first order predicate language. State what it means to say that S is consistent.

   (d) State and prove the Compactness Theorem for first order predicate logic.
   (You may assume the following form of the Completeness Theorem for first order predicate logic: If S is a set of sentences in a first order predicate language, then S is consistent if and only if S has a model.)

5. (a) Define the notion of a register machine, giving also a description of what a program is and of the types of instructions associated to the states of a program.

   (b) State what it means for a function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ to be computable.

   (c) Show that the following functions are computable:
   (i) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,,\quad f(m,n) = m + n$
   (ii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,,\quad f(m,n) = m + n + 2$
   (iii) $f : \mathbb{N}_0 \to \mathbb{N}_0\,,\quad f(m) = 3m$